As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Prove the “reverse triangle inequality”: show that in a normed linear space
\[ \|x\| - \|y\| \leq \|x - y\|. \]
What can we say about \(|\|x\| - \|y\||\) in relation to \(\|x - y\|\)?

3. Let \(\|\cdot\|_p\) be the \(\ell_p\) norm for vectors in \(\mathbb{R}^N\) as defined on page I.47 of the notes.
   (a) Prove that for any integer \(p \geq 1\)
   \[ \|x\|_p \leq \|x\|_1 \quad \text{for all } x \in \mathbb{R}^N \]
   (b) Prove that for any \(1 \leq q \leq p \leq \infty\)
   \[ \|x\|_p \leq \|x\|_q \quad \text{for all } x \in \mathbb{R}^N. \]

Hint: It is a fact that if \(a_1\) and \(a_2\) are non-negative real numbers, then
\[ (a_1 + a_2)\alpha \geq a_1\alpha + a_2\alpha \quad \text{for } \alpha \geq 1. \]

(c) Optional bonus question: Prove the fact in the hint above.

4. One way to visualize a norm in \(\mathbb{R}^2\) is by its unit ball, the set of all vectors such that \(\|x\| \leq 1\). For example, we have seen that the unit balls for the \(\ell_1, \ell_2,\) and \(\ell_\infty\) norms look like:

Given an appropriate subset of the plane, \(B \subset \mathbb{R}^2\), it might be possible to define a corresponding norm using
\[ \|x\|_B = \text{minimum value } r \geq 0 \text{ such that } x \in rB, \]
where \(rB\) is just a scaling of the set \(B:\)
\[ x \in B \implies r \cdot x \in rB. \]
\[ B_1 = \{ x : \| x \|_1 \leq 1 \} \quad B_2 = \{ x : \| x \|_2 \leq 1 \} \quad B_\infty = \{ x : \| x \|_\infty \leq 1 \} \]

(a) Let \( x = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \). For \( p = 1, 2, \infty \), find \( r = \| x \|_p \), and sketch \( x \) and \( rB_p \) (use different axes for each of the three values of \( p \)).

(b) Consider the 5 shapes below.

Explain why \( \| \cdot \|_{B_b} \) and \( \| \cdot \|_{B_c} \) are not valid norms. The most convincing way to do this is to find vectors for which one of the three properties of a valid norm are violated.

(c) Give a concrete method for computing \( \| x \|_{B_a}, \| x \|_{B_d}, \) and \( \| x \|_{B_e} \) for any given vector \( x \). (For example: for \( B_1 \), which corresponds to the \( \ell_1 \) norm, we would write \( \| x \|_1 = |x_1| + |x_2| \).) Using your expressions, show that these are indeed valid norms. This will require a little bit of thought.

5. Below, \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( \mathbb{R}^N \).

(a) Prove that \( |\langle x, y \rangle| \leq \| x \|_\infty \cdot \| y \|_1 \).

(b) Prove that \( \| x \|_1 \leq \sqrt{N} \cdot \| x \|_2 \). (Hint: Cauchy-Schwarz)
(c) Let $B_2$ be the unit ball for the $\ell_2$ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $y$:

$$\max_{x \in B_2} \langle x, y \rangle = ???$$

Describe the vector $x$ which achieves the maximum. (Hint: Cauchy-Schwarz)

(d) Let $B_\infty$ be the unit ball for the $\ell_\infty$ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $y$:

$$\max_{x \in B_\infty} \langle x, y \rangle = ???$$

Describe the vector $x$ which achieves the maximum. (Hint: Part (a))

(e) Let $B_1$ be the unit ball for the $\ell_1$ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $y$:

$$\max_{x \in B_1} \langle x, y \rangle = ???$$

Describe the vector $x$ which achieves the maximum. (Hint: Part (a))

(These last two might require some thought. If you solve them for $N = 2$, it should be easy to generalize.)

6. Let $A$ be the $2 \times 2$ matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 & 4 \\ -1/2 & 1/2 \end{bmatrix}.$$

For $x \in \mathbb{R}^2$, define $\|x\|_A = \|Ax\|_2$.

(a) Show that $\| \cdot \|_A$ is indeed a valid norm.

(b) Sketch the unit ball $B_A = \{ x : \|x\|_A \leq 1 \}$ corresponding to $\| \cdot \|_A$. Feel free to use MATLAB.