1. Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Suppose that $C$ and $D$ are closed convex sets. Show that $C = D$ if and only if their support functionals are equal to one another.

3. Let $S^N$ denote the space of $N \times N$ real-valued symmetric matrices with inner product $\langle X, Y \rangle = \text{trace}(XY)$. Of course this inner product obeys the standard Cauchy-Schwarz inequality:

$$\langle X, Y \rangle \leq \|X\|_F \|Y\|_F,$$

with equality if and only if $Y = \alpha X$. If $\lambda : S^N \to \mathbb{R}^N$ is the map from a symmetric matrix to its eigenvalues sorted in descending order,

$$\lambda(X) = \begin{bmatrix} \lambda_1(X) \\
\lambda_2(X) \\
\vdots \\
\lambda_N(X) \end{bmatrix}, \quad \lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda(X),$$

then we can re-write (1) as

$$\langle X, Y \rangle \leq \left( \sum_{n=1}^N \lambda_n^2(X) \right)^{1/2} \left( \sum_{n=1}^N \lambda_n^2(Y) \right)^{1/2},$$

since the Frobenius norm of a symmetric matrix is the same as the $\ell_2$ norm of its eigenvalues.

There is a more refined inequality for symmetric matrices, called the Fan inequality, which says

$$\langle X, Y \rangle \leq \langle \lambda(X), \lambda(Y) \rangle,$$

with equality if and only if $X$ and $Y$ are diagonalized with the same ordering by the same orthonormal eigenvectors $\{v_n\}$,

$$X = \sum_{n=1}^N \lambda_n(X) v_n v_n^T, \quad Y = \sum_{n=1}^N \lambda_n(Y) v_n v_n^T.$$
Note that this requires not only that $X$ and $Y$ have the same eigenvectors $v_1, \ldots, v_N$, but also that $v_n$ corresponds to the $n$th largest eigenvalue in both cases. Obviously, I am telling you all of this as it will be useful for solving the following:

(a) Let $C$ be the set of symmetric matrices with operator norm less than 1:

$$C = \{ X \in S^N : \| X \| \leq 1 \}, \quad \| X \| = \max_{\| v \|_2 = 1} \| X v \|_2.$$ 

Calculate the support functional

$$h(A) = \sup_{X \in C} \langle X, A \rangle.$$ 

(b) Given a $X_0 \in S^N$, describe how to solve the following closest point problem:

$$\min_{\| X \| \leq 1} \| X - X_0 \|_F.$$ 

(c) Define $\| X \|_* = \sum_{n=1}^N |\lambda_n(X)|$. Given a $X_0 \in S^N$, describe how to solve the following closest point problem:

$$\min_{\| X \|_* \leq 1} \| X - X_0 \|_F.$$ 

(d) Use the Fan inequality to show that the solution to the nonconvex problem

$$\min_{X \in S^N} \| X - X_0 \|_F \quad \text{subject to} \quad \text{rank}(X) = r,$$

is given by the truncated eigenvalue expansion for $X_0$.

(e) Discuss the similarities between the solutions in part (c) and part (d). What are the errors in both cases?

(f) Optional (and not easy): Prove the Fan inequality.

4. A sublevel set for a functional $f : \mathbb{R}^N \to \mathbb{R}$ is

$$S(f, \beta) = \{ x \in \mathbb{R}^N : f(x) \leq \beta \}.$$ 

(a) Show that if $f$ is a convex function, then all of its sublevel sets are convex.

(b) Show that the converse is not true by providing a counter-example (a function which has convex sublevel sets but is not convex).

5. (a) Prove the Jensen inequality; if $f$ is convex, then

$$f \left( \sum_i \theta_i x_i \right) \leq \sum_i \theta_i f(x_i), \quad \text{for all} \; \theta_i \text{ with } \sum_i \theta_i = 1, \; \theta_i \geq 0.$$
(b) Show that for $x_1, x_2 \geq 0$, arithmetic mean dominates geometric mean,

$$\sqrt{x_1 x_2} \leq \frac{x_1 + x_2}{2},$$

and more generally that

$$x_1^{\theta_1} x_2^{\theta_2} \cdots x_N^{\theta_N} \leq \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_N x_N,$$

for $x \in \mathbb{R}_+^N$, $\theta_i \geq 0$, and $\sum_n \theta_n = 1$.

(c) Show that for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\langle x, y \rangle \leq \|x\|_p \|y\|_q,$$

where $\|x\|_p = (\sum_n |x_n|^p)^{1/p}$.

(Hint: start with the case $\|x\|_p = 1$, $\|y\|_q = 1$ and then use homogeneity.)

6. Suppose I tell you that $X$ is a random vector with a covariance matrix of the form

$$E[XX^T] = \begin{bmatrix}
? & -1 & 5 & -4 & -4 \\
-1 & ? & 4 & 5 & 0 \\
5 & 4 & ? & -1 & -2 \\
-4 & 5 & -1 & ? & 1 \\
-4 & 0 & -2 & 1 & ?
\end{bmatrix}$$

Find the diagonal for this matrix that minimizes the maximum of the variances of the $X_i$. 